Name: L'Hospital

Calculus I

Professor Piotr Hajłasz

Second Exam

November 7, 2014

Problem	Possible points	Score
1	20	20
2	30	30
3	10	10
4	10	10
5	20	20
6	20	20
Total	110	110

To get A you only need 100 points, so 10 points is a bonus. In other words, if you miss 10 points on the exam you still get a full score.

I am sorry for the problem #2
I really should not do that.

P. Hajłasz

Problem 1. (20p) Find the absolute maximum and minimum values of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

Endpoints

$f(1) = 10(2 - \ln 1) = 10(2-0) = 20$

$f(e^2) = 10e^2(2 - \ln e^2) = 10e^2(2 - 2\ln e)$

$= 10e^2(2-2) = 0.$

$f(1) = 20, \quad f(e^2) = 0$

Critical points

$f'(x) = 10(2 - \ln x) + 10x\left(-\frac{1}{x}\right)$

$= 20 - 10\ln x - 10$

$= 10 - 10\ln x = 10(1 - \ln x)$

$f'(x) = 0 \iff \ln x = 1 \iff x = e$

$f(e) = 10e(2 - \ln e) = 10e(2-1) = 10e$

$f(e) = 10e$

$0 < 20 < 10e$

$f(e^2) < f(1) < f(e)$

$$\boxed{\begin{array}{l}\text{Absolute maximum } f(e) = 10e \\ \text{Absolute minimum } f(e^2) = 0\end{array}}$$

Problem 2. (30p=10+10+10p) Evaluate the limits

(a)

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right).$$

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\frac{\sin x - x}{x\sin x} \underset{\text{"}\frac{0}{0}\text{"}}{=}$$

$$\lim_{x\to 0}\frac{\cos x - 1}{\sin x + x\cos x} \underset{\text{"}\frac{0}{0}\text{"}}{=}$$

$$\lim_{x\to 0}\frac{-\sin x}{\cos x+\cos x - x\sin x} = \frac{0}{2} = \boxed{0}$$

Dear Dr. Hajłasz,
Really? Problem 2
should not appear on ~~my~~ exam!

(b)

$$\lim_{x\to 0^+} \frac{\ln x}{\ln(e^x - 1)}.$$

$$\lim_{x\to 0^+} \frac{\ln x}{\ln(e^x-1)} \underset{\text{"}\frac{-\infty}{-\infty}\text{"}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{e^x-1}\cdot e^x}$$

$$= \lim_{x\to 0^+} \frac{e^x-1}{x e^x} \underset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0^+} \frac{e^x}{e^x + x e^x} = \frac{1}{1+0}$$

$$= \boxed{1}$$

(c)

$$\lim_{x\to(\pi/2)^+} e^{\left(\tan x-\frac{1}{\cos x}\right)}.$$

$y = e^x$ is continuous and hence

$$\lim_{x\to(\pi/2)^+} e^{(\tan x - \frac{1}{\cos x})} = e^{\lim_{x\to(\pi/2)^+}(\tan x - \frac{1}{\cos x})}$$

$$\lim_{x\to(\pi/2)^+}\left(\tan x - \frac{1}{\cos x}\right) = \lim_{x\to(\pi/2)^+} \frac{\sin x - 1}{\cos x} \overset{\text{"}\frac{0}{0}\text{"}}{=}$$

$$= \lim_{x\to(\pi/2)^+} \frac{\cos x}{-\sin x} = \frac{0}{-1} = 0$$

$$\lim_{x\to(\pi/2)^+} e^{(\tan x - \frac{1}{\cos x})} = e^0 = \boxed{1}$$

Problem 3. (10p) Find the inverse of $f(x) = e^x - e^{-x}$. **Hint:** *At some point replacing* e^x *by* z *will lead to a quadratic equation in* z. *Since* $z = e^x > 0$, *only one solution will be acceptable.*

$y = e^x - e^{-x}$. Solve for x.

$y e^x = e^{2x} - 1$

$e^{2x} - y e^x - 1 = 0$. Let $z = e^x$.

$z^2 - yz - 1 = 0$

$$z_{1,2} = \frac{y \pm \sqrt{y^2+4}}{2}$$

Since $z = e^x > 0$ only the "+" solution is okay ("−" solution is negative)

$$e^x = z = \frac{y + \sqrt{y^2+4}}{2}$$

$$x = \ln\left(\frac{y + \sqrt{y^2+4}}{2}\right)$$

and finally

$$\boxed{f^{-1}(x) = \ln\left(\frac{x + \sqrt{x^2+4}}{2}\right).}$$

Problem 4. (10p=5+5p) Using Newton's method for the approximation of the solution to $e^{-x} = x - 2$:

(a) Find the general formula for x_{n+1} in terms of x_n.

We are looking for a zero of the function

$$f(x) = e^{-x} - x + 2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{e^{-x_n} - x_n + 2}{-e^{-x_n} - 1}$$

$$\boxed{x_{n+1} = x_n + \frac{e^{-x_n} - x_n + 2}{e^{-x_n} + 1}}$$

(b) Find x_2 if $x_1 = 1$. Simplify the answer.

$$x_2 = 1 + \frac{e^{-1} - 1 + 2}{e^{-1} + 1} = 1 + \frac{e^{-1} + 1}{e^{-1} + 1} = 2$$

$$\boxed{x_2 = 2}$$

Problem 5. (20p) Find the point on the line $\frac{x}{a} + \frac{y}{b} = 1$, that is closest to the origin.

$\frac{x}{a} + \frac{y}{b} = 1$, $\frac{y}{b} = 1 - \frac{x}{a}$, $y = b(1 - \frac{x}{a})$.

Points on the line are $(x, b(1 - \frac{x}{a}))$.

We want to find the point where the distance to the origin attains minimum

$$d(x) = \sqrt{x^2 + b^2(1 - \frac{x}{a})^2}$$

Clearly d attains a minimum (that follows from geometric considerations) and it is attained at the same point at the minimum of

$$f(x) = d(x)^2 = x^2 + b^2(1 - \frac{x}{a})^2.$$

Clearly f attains minimum at a critical point

$$f'(x) = 2x + 2b^2(1 - \frac{x}{a})(-\frac{1}{a}) = 0$$

$$x - \frac{b^2}{a}(1 - \frac{x}{a}) = 0$$

$$x - \frac{b^2}{a} + \frac{b^2 x}{a^2} = 0$$

$$x(1 + \frac{b^2}{a^2}) = \frac{b^2}{a}, \quad x\left(\frac{a^2+b^2}{a^2}\right) = \frac{b^2}{a}$$

$$x = \frac{a^2}{a^2+b^2} \frac{b^2}{a} = \frac{ab^2}{a^2+b^2}.$$

$$y = b(1 - \frac{x}{a}) = b(1 - \frac{b^2}{a^2+b^2}) = b\frac{a^2+b^2-b^2}{a^2+b^2} = \frac{ba^2}{a^2+b^2}.$$

The minimum distance to the origin is attained at the point

$$(x, y) = \left(\frac{ab^2}{a^2+b^2}, \frac{ba^2}{a^2+b^2}\right).$$

Problem 6. (20p) Sketch the graph of the function $f(x) = x^4 - 4x^3 + 10$. Make sure that you clearly label: intervals where the function in increasing, decreasing, concave up and concave down, local and absolute maxima/minima and inflection points.

$$f(x) = x^4 - 4x^3 + 10$$
$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

f'	$-$		$-$		$+$
f	dec	0	dec	3	inc

$x=0$, $x=3$ critical points
absolute minimum at $x=3$

f''	$+$		$-$		$+$
f	$\cup$	0	$\cap$	2	$\cup$

$x=0$, $x=2$
inflection points

$f(0)=10$, $f(2)=-6$, $f(3)=-17$